Statistics of unstable periodic orbits of a chaotic dynamical system with a large number of degrees of freedom

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For a simple model of chaotic dynamical systems with a large number of degrees of freedom, we find that there is an ensemble of unstable periodic orbits (UPOs) with the special property that the expectation values of macroscopic quantities can be calculated using only one UPO sampled from the ensemble. Evidence to support this conclusion is obtained by generating the ensemble by Monte Carlo calculation for a statistical mechanical model described by a space-time Hamiltonian that is expressed in terms of Floquet exponents of UPOs. This result allows us to interpret the recent interesting discovery that statistical properties of turbulence can be obtained from only one UPO [G. Kawahara and S. Kida, J. Fluid Mech. **449**, 291 (2001); S. Kato and M. Yamada, Phys. Rev. E **68**, 025302(R) (2003)].

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A wide range of systems, including fluid turbulence and ecosystems, can be described by chaotic dynamical systems (CDSs) with a large number of degrees of freedom. Because the evolution equations describing them are nonlinear and possess many degrees of freedom, analyses employing various theoretical tools fail. For instance, it is difficult to characterize quantitatively the intermittency of fluid turbulence using perturbative expansion methods applied to the Navier-Stokes equation [1].

In contrast, for CDSs consisting of assemblies of molecules at equilibrium, equilibrium statistical mechanics provides a powerful framework to predict macroscopic properties without the need to analyze a Hamiltonian equation for a large number of molecules. The success of equilibrium statistical mechanics relies on its probabilistic description: In order to predict macroscopic properties, the exact probability distribution for states is not needed. Rather, the existence of many degrees of freedom allows a tractable distribution to reproduce macroscopic properties correctly. When considering macroscopic properties of a CDS with many degrees of freedom, it is tempting to think that a probabilistic approach other than analyzing evolution equations can be employed by finding a useful probability measure for the system.

Although it would be extremely difficult to find a useful measure for general CDSs, it was suggested recently that an ensemble of states that describes macroscopic properties of CDSs can be constructed from a special unstable periodic orbit (UPO) [2,3]. The first study of such a UPO demonstrated that the spatial profiles of the mean and variance of the velocity in minimal wall turbulence can be extracted from only one UPO of the Navier-Stokes equation [2]. Subsequently, the scaling exponents of velocity fluctuations, which characterize the intermittency of a turbulent velocity field, was found to be obtained from only one UPO in the GOY shell model [3].

To obtain both of these interesting results, the UPOs were found in numerical computations such as the Newton-Raphson method applied to a function of 15 422 variables [2] and following the destabilization of a limit cycle resulting from a bifurcation process in the GOY model with 24 degrees of freedom [3]. Because there are other UPOs which yield properties that differ from those of the special UPOs, it can be considered that these special UPOs were selected out of infinitely many UPOs under some criterion. However, such a criterion for choosing UPOs is still unknown, because in the previous works the initial value for the Newton-Raphson method and the path of the destabilization of the limit cycle were found by trial and error. Given this situation, the objective of the present work is to find the criterion to choose special UPOs such that macroscopic properties can be obtained with high accuracy even when only one UPO element of the ensemble is used.

In this Brief Report, we present analysis in which such a UPO ensemble is obtained for a CDS with many degrees of freedom. Note that here, the term "macroscopic quantities" refers to quantities obtained by taking the average over many degrees of freedom of the system. Furthermore, the term "macroscopic properties" is used to mean the leading terms of the expectation values of the macroscopic quantities with respect to the natural invariant measure when the number of degrees of freedom is large.

A. Model. Here, we explain the model analyzed in the present work. First, we note that it is not necessary to study turbulence for our present purpose. Rather, it may be preferable to analyze a simple model for which many UPOs can be found easily. With this idea, we study a coupled map lattice (CML) proposed by Sakaguchi [4], which we now describe. Let $(x_i, \Delta_i) \in [-1, 1] \times [-1, 1]$ be dynamical variables defined on the ith site of a one-dimensional lattice consisting of N lattice points numbered from 0 to $N-1$ [5]. For later conve-

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nience, the variable s^t_i, which is called the "spin," is defined as

$$s_i \equiv \begin{cases} +1 & (-1 \leq x_i < \Delta_i), \\ -1 & (\Delta_i \leq x_i \leq 1). \end{cases} \quad (1)$$

The time evolution of (x_i, Δ_i) is given by

$$(x^{t+1}_i, \Delta^{t+1}_i) = \left(f(x^t_i, \Delta^t_i), \tanh\left[\frac{k}{2}(s^t_{i-1} + s^t_{i+1})\right] \right) \quad (2)$$

for $i = t+1 \pmod 2$ and $(x^{t+1}_i, \Delta^{t+1}_i) = (x^t_i, \Delta^t_i)$ for $i = t \pmod 2$. Here k is a positive parameter, and the local map $f(x^t_i, \Delta^t_i)$ is the Bernoulli map, given by

$$f(x^t_i, \Delta^t_i) \equiv \frac{2(x^t_i + s^t_i)}{1 + s^t_i \Delta^t_i} - s^t_i. \quad (3)$$

Note that the discontinuous point of the map $f(x, \Delta)$ is located at $x = \Delta$, and the spin variable, s_i, is identified to a symbol of the symbolic dynamical system for the local map $f(x_i, \Delta_i)$. Hereafter, we call the above CML the "Bernoulli CML" and denote it $X^{t+1} = F(X^t)$, where $X^t \equiv \{x^t_i, \Delta^t_i\}^{N-1}_{i=0}$.

The Bernoulli CML has interesting features, as demonstrated by Sakaguchi [4]. First, the natural invariant measure for spin configurations coincides with the canonical distribution for an Ising spin Hamiltonian. In order to express this fact explicitly, we define $J(s)$ as the set of states $\{x_i, \Delta_i\}^{N-1}_{i=0}$ corresponding to the spin configuration $s \equiv \{s_i\}^{N-1}_{i=0}$. Then, it is found that the natural invariant measure on the set $J(s)$ can be written as $\mu(J(s)) = \text{const} \times \exp[(k/2)\Sigma^{N-1}_{i=0} s_i s_{i+1}]$. Furthermore, when the initial probability distribution of states is the natural invariant measure, it is known that the transition probability $T(s|s')$ from one spin configuration s to another s' is given by $T(s^{2t}|s^{2t+1}) = \Pi^{N/2-1}_{i=0}(1 + s^{2t+1}_{2i+1}\Delta^{2t+1}_{2i+1})/2$, $T(s^{2t+1}|s^{2t+2}) = \Pi^{N/2-1}_{i=0}(1 + s^{2t+2}_{2i}\Delta^{2t+2}_{2i})/2$.

B. A UPO ensemble. We now proceed to construct a UPO ensemble for the Bernoulli CML with the property that the expectation values of macroscopic quantities are obtained with high accuracy even when only one UPO element of the ensemble is used. As the first step in this construction, we demonstrate that there is a one-to-one correspondence between symbol sequences and UPOs. Suppose that a symbol sequence $[s] \equiv (s^0, s^1, \ldots, s^{p-1})$ is given. We then attempt to find a periodic point $X^0 \equiv \{x^0_i, \Delta^0_i\}^{N-1}_{i=0}$ corresponding to $[s]$. The N-tuple $\{\Delta^0_i\}^{N-1}_{i=0}$ can be determined directly from the given symbol sequence by definition. A component of $X^p \equiv F^p(X^0)$ is given by $x^p_i = \Pi^{p/2-1}_{t=0} a^{2t}_i x^0_i + \Sigma^{p/2-1}_{t'=1}\Pi^{p/2-1}_{t=t'} a^{2t}_i s^{2t'-2}_i(a^{2t'-2}_i - 1) + s^{p-2}_i(a^{p-2}_i - 1)$, where it is assumed that p is an even number and $a^t_i \equiv 2/(1 + s^t_i \Delta^t_i)$. Because $\Pi^{p/2-1}_{t=0} a^t_i \neq 1$ for an arbitrary symbol sequence, we can uniquely determine the point satisfying the periodicity condition $x^0_i = x^p_i$. In this way, we can find a point $X^0 \equiv \{x^0_i, \Delta^0_i\}^{N-1}_{i=0}$ such that $X^0 = X^p$ for a given symbol sequence $[s]$. Conversely, it should be confirmed that the symbol sequence generated from the point X^0 coincides with $[s]$. We numerically confirmed this for 10^6 symbol sequences generated randomly. On the basis of these results, we conclude

that there exists a single UPO corresponding to an arbitrary symbol sequence.

Because of the one-to-one correspondence between UPOs and symbol sequences, we can obtain a UPO ensemble through the construction of an ensemble of symbol sequences. One natural possibility for a probability measure (PM) on symbol sequences is the frequency distribution of symbol sequences in the case that the initial probability distribution of states is given by the natural invariant measure. The frequency distribution of a symbol sequence $[s]$ is expressed as $P([s]) = \mu(J(s^0))T(s^0|s^1)T(s^1|s^2)\cdots T(s^{p-1}|s^p)$. Although this expression appears to take a simple form, it is not easy to derive the natural invariant measure for most dynamical systems. In order to analyze a wide variety of dynamical systems, it is convenient to use a more tractable PM that yields the same macroscopic properties as those obtained from $P([s])$. As one such possibility, we consider the PM $Q([s])$ obtained by replacing the natural invariant measure μ in $P([s])$ with a constant Z^{-1}. This PM can be rewritten as

$$Q([s]) = \exp[-\ln|D^{(u)}F^p(X^0)|]/Z. \quad (4)$$

Here, $|D^{(u)}F^p(X^0)|$ is the absolute value of the product of the eigenvalues in the expanding directions of the Jacobi matrix for the map given by p iterations of the coupled map F evaluated at the periodic point X^0 corresponding to the symbol sequence $[s]$, and Z is a normalization constant. The expression of frequency distribution of a symbol sequence has been obtained in Ref. [6]. The PM defined by Eq. (4) can be calculated more easily for any CDS than that defined by $P([s])$.

C. Numerical demonstration. Now we describe the numerical demonstration that a macroscopic property for the CML can be determined with high accuracy even when only one UPO element of the ensemble determined by Eq. (4) is used to provide ensembles of states. First, we note that the sampling of UPOs according to Eq. (4) can be carried out using the Monte Carlo method. In this procedure, a symbol sequence $[s]$ is regarded as a spin configuration on a $(1+1)$-dimensional $p \times N$ lattice, where p and N correspond to the time and space directions, respectively. In addition, the PM given by Eq. (4) is of the same form as the canonical distribution. Hence, the UPO ensemble we seek can be generated with the Metropolis algorithm, regarding $\ln|D^{(u)}F^p(X^0)|$ as the "Hamiltonian" of a $(1+1)$-dimensional Ising model with spins $[s]$. In order to obtain UPOs, periodic boundary conditions are imposed at the boundaries of the lattice in the time direction, $t = 0$ and $t = p-1$.

For each UPO of period p, an ensemble of spin configurations s, which can be regarded as an ensemble of states, can be constructed by assuming the equal weight probability $1/p$ on each spin configuration of the sequence $[s]$ corresponding to the UPO. Using this ensemble associated with one UPO, we consider the ensemble average of the macroscopic quantity $-\Sigma^{N-1}_{i=0} s_i s_{i+1}/N$, given by $\varepsilon_N \equiv -\Sigma^{p-1}_{t=0}\Sigma^{N-1}_{i=0} s^t_i s^t_{i+1}/(Np)$. In Fig. 1, the average and standard deviation of ε_N calculated from 50000 samples of UPOs generated by the Monte Carlo method are displayed for several
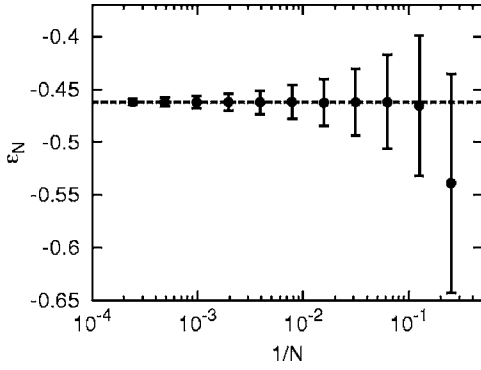
FIG. 1. Averages (solid circles) and standard deviations (error bars) of ε_N evaluated from 50 000 samples of UPOs for the case $N=2^n (2 \leq n \leq 12)$, $k=1$, and $p=64$. The broken line represents the exact value in the limit $N \rightarrow \infty$, $-\tanh(1/2)$.

values of N. It is seen that as N increases, the standard deviation approaches zero and the average value of ε_N converges to $-\tanh(k/2)$, which is the value calculated from the natural invariant measure $\mu(J(s))$. This implies that in the limit $N \rightarrow \infty$, the expectation value of $-\Sigma_{i=0}^{N-1} s_i s_{i+1}/N$ with respect to the natural invariant measure can be calculated from the ensemble of states obtained from only one UPO, because the discrepancy between the value calculated from one UPO and that from another UPO can be ignored.

D. Single-UPO description. We consider here the mechanism to make it possible to describe macroscopic properties using only one UPO. First, let us consider relaxation processes to the steady state provided that a smooth initial measure is given. Then, the expectation value of a quantity that is a function of a symbol, $A(s)$, converges to the same value in a relaxation time τ_A, irrespective of initial measures. Thus, the asymptotic value for the case that the initial measure is assumed to be flat, where the frequency distribution of the symbol sequence is equal to $Q([s])$, coincides with the value for the case that the initial measure is given by the natural measure μ, whose value is nothing but the expectation value of $A(s)$ with respect to μ. This is expressed simply as $\langle A(s^t) \rangle_Q \equiv \Sigma_{[s]} Q([s]) A(s^t) \simeq \langle A(s) \rangle_\mu \equiv \Sigma_s \mu(J(s)) A(s)$ in the time range $t > \tau_A$.

Next, by using this generic property, we can estimate the expectation value of the time average of $A(s)$ as $\langle \Sigma_{t=0}^{p-1} A(s^t)/p \rangle_Q \simeq \Sigma_{t=0}^{\tau_A - 1} \langle A(s^t) \rangle_Q/p + (p - \tau_A) \langle A(s) \rangle_\mu/p$. When the period p is taken to be long compared to the relaxation time τ_A, we obtain $\langle \Sigma_{t=0}^{p-1} A(s^t)/p \rangle_Q \simeq \langle A(s) \rangle_\mu$.

Finally, we consider the case where $A(s)$ is a macroscopic quantity such as ε_N. The law of large numbers asserts that fluctuations of such a quantity become negligible when the degrees of freedom is large and steady state is settled. Hence, for almost all symbol sequence $[s]$ sampled according to $Q([s])$, $\Sigma_{t=0}^{p-1} A(s^t)/p \simeq \langle \Sigma_{t=0}^{p-1} A(s^t)/p \rangle_Q$.

Consequently, $\Sigma_{t=0}^{p-1} A(s^t)/p \simeq \langle A(s) \rangle_\mu$ for almost all $[s]$ with respect to $Q([s])$. Because of one-to-one correspondence between UPOs and symbol sequences, the time average along single UPO sampled according to $Q([s])$ provides a good approximate value of $\langle A_N(s) \rangle_\mu$. The discussion presented above shows that many degrees of freedom and the

law of large numbers applied to macroscopic quantities are essential for the single-UPO description.

E. Interpretation of the UPO description of turbulence. We now interpret the UPO description of turbulence [2,3] based on our result. In particular, we discuss the relationship between the UPOs employed in Refs. [2,3] and the UPOs sampled according to Eq. (4).

In the methods of searching for UPOs employed in the two previous works, a trial initial state point is first chosen near a true periodic point X^0 in some way, and the time development of the state is traced by numerical integration of the governing equation. Then, the initial point is recognized as a periodic point if the distance between the orbit and the trial initial state point at the time that the orbit returns closest to the trial initial state point is less than some threshold value δ.

For simplicity, suppose that a set of trial initial state points are scattered within a hypersphere of radius δ_0 centered at a periodic point X^0 of period p. Then, this set expands in the unstable direction as the system evolves in time. Specifically, assuming that d_u is the dimension of the unstable manifold and δ_0 is sufficiently small, the volume of the set in the unstable direction becomes $\delta_0^{d_u} |D^{(u)} F^p(X^0)|$ after one period of motion along the UPO starting from X^0. Hence, the probability that the distance between the orbit and the trial initial state point after one period of motion remains less than the threshold δ is estimated as $\delta^{d_u}/(\delta_0^{d_u} |D^{(u)} F^p(X^0)|)$. This implies that the probability of finding numerically an approximate periodic point around X^0 is proportional to $|D^{(u)} F^p(X^0)|^{-1}$, which is equal to the quantity appearing in Eq. (4). From these considerations, we conjecture that the UPOs found in the previous works [2,3] were sampled from a probability measure that takes a form similar to Eq. (4) for turbulence.

In order to demonstrate the correspondence between the single-UPO description of turbulence and our result that the expectation value of a quantity averaged over many degrees of freedom can be obtained from only one UPO, one task remains: We must show that the quantities analyzed in Refs. [2,3] were also averaged over many degrees of freedom. First, the quantities investigated in the case of wall turbulence [2] were averaged on a plane parallel to the walls. Next, consider whether the scaling exponents of the turbulent velocity field analyzed in Ref. [3] can be expressed in an averaged form over many degrees of freedom. By making use of notions on the energy cascade and self-similarity in the inertial subrange, the scaling exponent of order q, ζ_q, can be derived as $\zeta_q = q/3 - \ln\langle(\epsilon_{j+1}/\epsilon_j)^{q/3}\rangle$, where ϵ_j is the energy dissipation rate at the jth shell [7]. Self-similarity makes the statistical average $\langle \cdot \rangle$ replaceable with the average over shells in the inertial subrange as $\langle(\epsilon_{j+1}/\epsilon_j)^{q/3}\rangle \simeq (1/N)\Sigma_{j=0}^{N-1}(\epsilon_{j+1}/\epsilon_j)^{q/3}$. Thus, the scaling exponent ζ_q can be expressed as a quantity averaged over identically distributed variables.

One may wonder whether the numbers of statistically independent variables in the systems studied in Refs. [2,3] are sufficiently large to ensure that the UPOs sampled according to Eq. (4) should actually to yield values close to the true

expectation value. However, as seen in Fig. 1, the results for the cases $N=8$ and 16 indeed provide good approximations of the true expectation value. Therefore, in the turbulence problems, we believe that the numbers of statistically independent variables are sufficient to provide good approximations.

F. Discussion. The previous works, Refs. [2,3], and the above interpretation show that a sample UPO chosen with the weight $|D^{(u)}F^p(X^0)|^{-1}$ can describe macroscopic properties of turbulence. It suggests that the single-UPO description has a wide range of application.

Let us discuss the interrelationship between the present work and other studies. The statistical analysis of symbol sequences for orbits in dynamical systems is called the "thermodynamic formalism" [8,9]. The Monte Carlo calculation of Eq. (4) can be regarded as a numerical realization of the thermodynamic formalism for the Bernoulli CML. In similar studies, the explicit construction of the space-time Hamiltonian for spatially extended dynamical systems was proposed in Ref. [10], though its construction is based on a natural invariant measure that happened to be obtained for specific models.

In hyperbolic dynamical systems, employing the periodic orbit expansion, the natural invariant measure μ on a set R in phase space can be expressed as $\mu(R) = \lim_{p \to \infty} \Sigma_j |D^{(u)}F^p(X_j)|^{-1}$, where the sum is taken over all the fixed points X_j of F^p, i.e., UPOs, in the set R [11–13]. In related studies, some CDSs with many degrees of freedom have been analyzed using the periodic orbit expansion technique [14–16]. From this expanded form, it turns out that the weight of a UPO is proportional to $\lim_{p \to \infty} |D^{(u)}F^p(X_j)|^{-1}$. Although this expression reminds us the form given in Eq. (4), the limit $p \to \infty$ shouled be taken to obtain the precise natural measure. In contrast to that, our results suggest that the periodic orbit expansion with a finite period is sufficient to describe macroscopic properties. Note that it is an important open qeustion to evaluate precisely the period p for the single-UPO description.

In summary, we have demonstrated that the macroscopic properties of the Bernoulli CML can be calculated with high accuracy using only one UPO sampled from the special UPO ensemble described by Eq. (4).

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